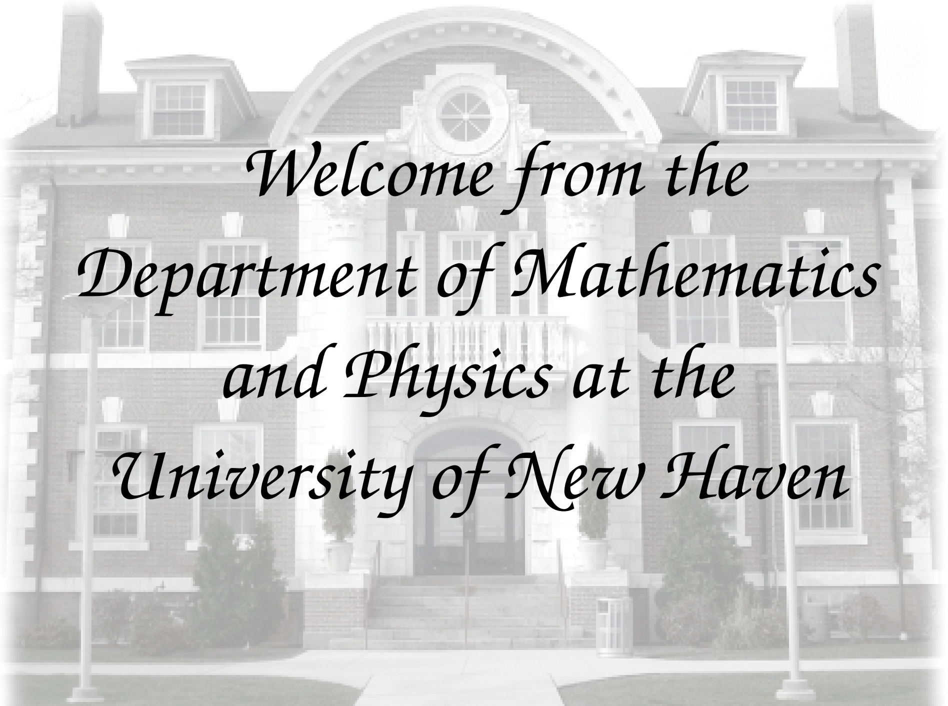

Welcome from the Department of Mathematics and Physics at the University of New Haven

Matrix Braids

Joseph Kolibal

The University of New Haven

October, 19 2017

Abstract

Braiding matrices arise as a subtopic of the the Yang-Baxter equation, which has been studied extensively due to application in numerous fields of mathematics and physics. We connect these to a simplified matrix representation and focus on obtaining solutions to matrix braids by considering special matrices where solutions are more easily found. Finally, we suggest a fixed point iteration algorithm to determine the braid complement of a given matrix, if it exists.

- We analyze the Yang-Baxter equation specialized to matrices $A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $B : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, having the following form

$$ABA = BAB. \quad (1)$$

- We seek to characterize solutions of (1), including finding the necessary and if possible sufficient conditions under which distinct matrices A and B satisfy (1).
- In that regard, the approach is not too dissimilar to analyzing the structure of $AB = BA$, i.e., determining when two distinct matrices commute.[1]

In that sense it seems appropriate to coin the usage that two distinct matrices form a braid, or more simply braid if they satisfy (1).

[1] The physics preamble suggests the form $ABA = BAB$, but why not consider when $AAB = BAB$, or $BAA = BAB$, or ..., i.e., consider permutations of two matrices takesn three at a time. As an academic pursuit, we can consider the permutation products of n matrices taken p at a time, but we can also consider matrix multiplication schemes for public key encryption, e.g., the Simple Matrix Scheme. The simpler Hill cipher was the first attempt to do cryptography with matrices.

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- An obvious necessary condition to have braiding matrices, i.e., to satisfy (1), is that

$$\det(A)\det(B)\det(A) = \det(B)\det(A)\det(B)$$

$$\Longrightarrow \det(A) = \det(B) \text{ if } A, B \text{ nonsingular} .$$

- Since the determinant of a matrix is equal to the product of the eigenvalues of the matrix, we have

$$\prod_i \lambda_i(A) = \prod_j \lambda_j(B),$$

where λ_i and λ_j are all the eigenvalues of A and B, including multiplicities.

Not much information if A or B are singular.

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- We can write the brading matrices in CFE form as

$$XA = BX, \tag{2}$$

where $X = AB$.

- Adding A to both sides of (2), we obtain $(X+I)A = BX + A$.
- Solving for A on the left side of the equation yields our fixed point iteration method

$$A = (X+I)^{-1}(BX + A). \tag{3}$$

Must ensure that $(X+I)$ remains invertible during the iteration.

- We could rewrite (3) as

$$A = (X+cI)^{-1}(BX + cA), \tag{4}$$

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- Example of using a table and turning of the background.

n	V_n/c_n	V_n/C_n	S_n	Comments
1	2.0000	1.000000	2.0000	Since $S_1 = \frac{2\pi^{1/2}}{\Gamma(1/2)} r^{1-1}$ and $\Gamma(1/2) = \pi^{1/2}$
2	3.1416	0.785398	6.2832	
3	4.1888	0.523599	12.566	
4	4.9348	0.308425	19.739	
5	5.2638	0.164493	26.319	Maximum V_n. Holds the most n-cubes.
6	5.1677	0.080746	31.006	
7	4.7248	0.036912	33.074	Maximum S_n at $n = 7.257\ldots$.
8	4.0587	0.015854	32.497	
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Note that $V_n/C_n \to 0$, i.e., spheres are vanishingly small inside of cubes in $\mathbb{R}^n$ for large n.

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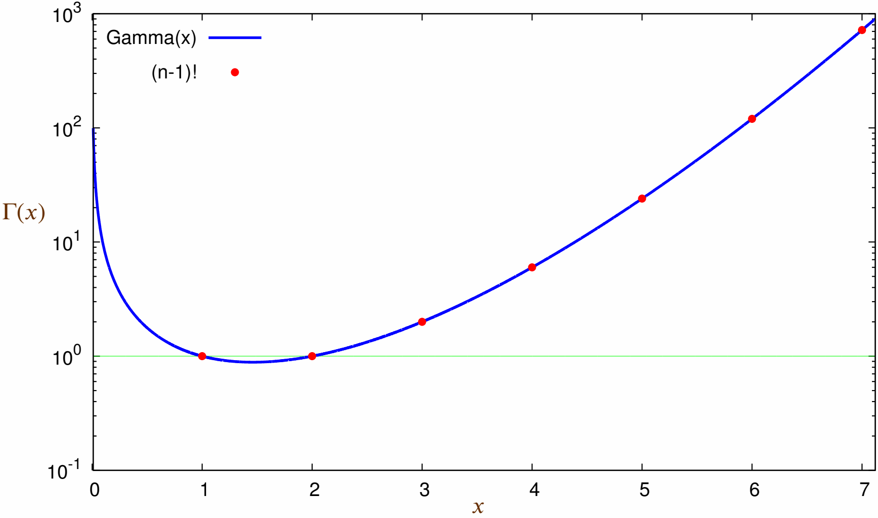

Figure: Plot of Gamma function, $\Gamma(x)$, showing factorials, $\Gamma(x) = (n-1)!$ for $x = 1, 2, \ldots$.

```
\begin{frame}
\Ft{Elementary considerations}
\begin{itemize}
\item
An obvious necessary condition to have braiding matrices, i.e., to
satisfy (\ref{eq:first}), is that
\begin{equation*}
\begin{aligned}
 \det(A)\det(B)\det(A)= &  \det(B)\det(A)\det(B) \\  \pause
 &  \implies \det(A) = \det(B)\mbox{ if } A, B \mbox{ nonsingular}.
\end{aligned}
\end{equation*}
\pause

\item
Since the \Urlx{https://wiki2.org/en/Determinant+Brights}{determinant}
of a matrix is equal to the product of the
Urlx{https://wiki2.org/en/Eigenvalues and eigenvectors+Brights}{eigenvalues}
of the matrix, we have
%
\begin{equation*}
 \prod_i{\lambda_{i}(A)} = \prod_j{\lambda_{j}(B)},
\end{equation*}
%
%
where $ \lambda_i$ and $ \lambda_j$ are all the eigenvalues of $ A$ and $ B$,
including multiplicities.
\vspace{0.8em}
\pause

{\color{\quotecolor}
\begin{center}
Not much information if $ A$ or $ B$ are singular.
\end{center}
}

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\end{frame}
```

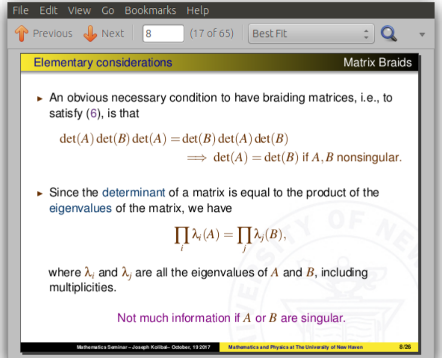

We used Beamer, a version of LaTeX that is highly optimized to produce quality presentation slides. Interested? Consider MATH 2212 Software Tools for Math, along with some self-help research tools. Note that we can invoke actions, such as view the jpg file that is the background for this page.

Back to Elementary Considerations, pg.8

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