Welcome from the Department of Mathematics and Physics at the University of New Haven

Matrix Braids

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Abstract

Braiding matrices arise as a subtopic of the the Yang-Baxter equation, which has been studied extensively due to application in numerous fields of mathematics and physics. We connect these to a simplified matrix representation and focus on obtaining solutions to matrix braids by considering special matrices where solutions are more easily found. Finally, we suggest a fixed point iteration algorithm to determine the braid complement of a given matrix, if it exists.

- We analyze the Yang-Baxter equation specialized to matrices $A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $B : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, having the following form

$$ABA = BAB. \tag{1}$$

- We seek to characterize solutions of (1), including finding the necessary and if possible sufficient conditions under which distinct matrices A and B satisfy (1).
- In that regard, the approach is not too dissimilar to analyzing the structure of $AB = BA$, i.e., determining when two distinct matrices commute.[1]

In that sense it seems appropriate to coin the usage that two distinct matrices form a braid, or more simply braid if they satisfy (1).

[1] The physics preamble suggests the form $ABA = BAB$, but why not consider when $AAB = BAB$, or $BAA = BAB$, or ..., i.e., consider permutations of two matrices takesn three at a time. As an academic pursuit, we can consider the permutation products of n matrices taken p at a time, but we can also consider matrix multiplication schemes for public key encryption, e.g., the Simple Matrix Scheme. The simpler Hill cipher was the first attempt to do cryptography with matrices.

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- An obvious necessary condition to have braiding matrices, i.e., to satisfy (1), is that

$$\det(A)\det(B)\det(A) = \det(B)\det(A)\det(B)$$

$$\implies \det(A) = \det(B) \textit{ if } A, B \textit{ nonsingular} .$$

- Since the determinant of a matrix is equal to the product of the eigenvalues of the matrix, we have

$$\prod_i \lambda_i(A) = \prod_j \lambda_j(B),$$

where λ_i and λ_j are all the eigenvalues of A and B, including multiplicities.

Not much information if A or B are singular

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- We can write the brading matrices in CFE form as

$$XA = BX, \tag{2}$$

where $X = AB$.

- Adding A to both sides of (2), we obtain $(X+I)A = BX + A$.
- Solving for A on the left side of the equation yields our fixed point iteration method

$$A = (X+I)^{-1}(BX+A). \tag{3}$$

Must ensure that $(X+I)$ remains invertible during the iteration.

- We could rewrite (3) as

$$A = (X+cI)^{-1}(BX+cA), \tag{4}$$

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- Example of using a table and turning of the background.

n	V_n/c_n	V_n/C_n	S_n	Comments
1	2.0000	1.000000	2.0000	Since $S_1 = \frac{2\pi^{1/2}}{\Gamma(1/2)} r^{1-1}$ and $\Gamma(1/2) = \pi^{1/2}$
2	3.1416	0.785398	6.2832	
3	4.1888	0.523599	12.566	
4	4.9348	0.308425	19.739	
5	5.2638	0.164493	26.319	Maximum V_n. Holds the most n-cubes.
6	5.1677	0.080746	31.006	
7	4.7248	0.036912	33.074	Maximum S_n at $n = 7.257\ldots$.
8	4.0587	0.015854	32.497	
9	3.2985	0.0064424	29.687	

Note that $V_n/C_n \to 0$, i.e., spheres are vanishingly small inside of cubes in $\mathbb{R}^n$ for large n.

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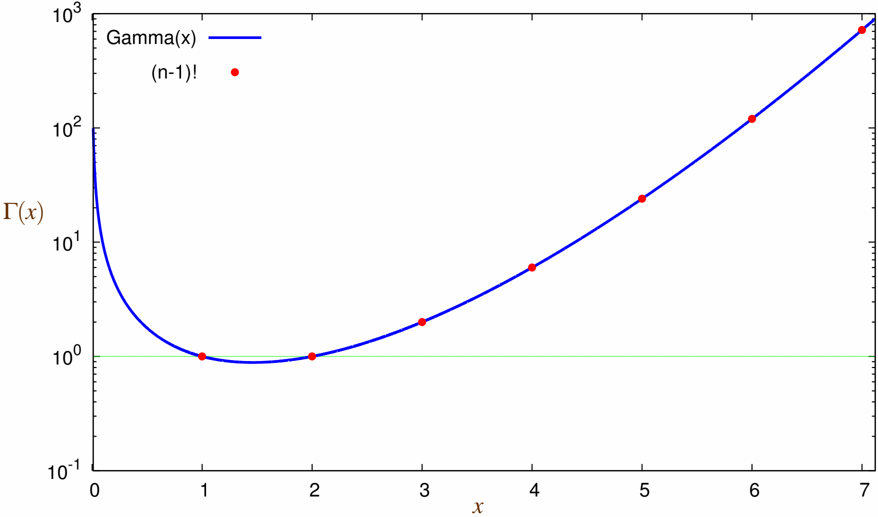


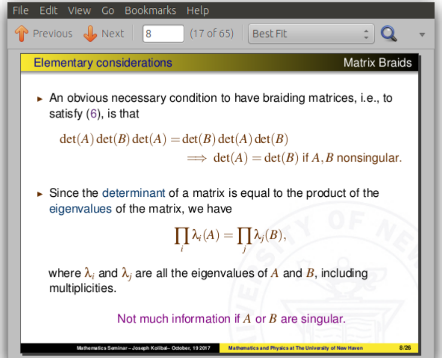
Figure: Plot of Gamma function, $\Gamma(x)$, showing factorials, $\Gamma(x) = (n-1)!$ for $x = 1, 2, \ldots$.

```
\begin{frame}
\Ft{Elementary considerations}
\begin{itemize}
\item
An obvious necessary condition to have braiding matrices, i.e., to
satisfy (\ref{eq:first}), is that
\begin{equation*}
\begin{aligned}
 \det(A)\det(B)\det(A)= &  \det(B)\det(A)\det(B) \\  \pause
 &  \implies \det(A) = \det(B)\mbox{ if } A, B \mbox{ nonsingular}.
\end{aligned}
\end{equation*}
\pause

\item
Since the \Urlx{https://wiki2.org/en/Determinant+Brights}{determinant}
of a matrix is equal to the product of the
Urlx{https://wiki2.org/en/Eigenvalues_and_eigenvectors+Brights}{eigenvalues}
of the matrix, we have
%
\begin{equation*}
 \prod_i{\lambda_{i}(A)} = \prod_j{\lambda_{j}(B)},
\end{equation*}
%
%
where $ \lambda_i$ and $ \lambda_j$ are all the eigenvalues of $ A$ and $ B$,
including multiplicities.
\vspace{0.8em}
\pause

{\color{\quotecolor}
\begin{center}
Not much information if $ A$ or $ B$ are singular.
\end{center}
}

\end{itemize}
\end{frame}
```



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Back to **Elementary Considerations**, pg.8

Thank You

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