Volumes, surfaces, and integrals, the Title of the Presentation

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Abstract

The relationship of integrals on the surface of a region in $\mathbb{R}^n$ to integrals over the volume of the region is a fundamental part of calculus. We examine this relationship from the fundamental theorem of calculus and integration by parts, to the theorems of vector calculus, taking note of some interesting aspects of the technique and the diversity of results which can be obtained using these relationships.

In particular, the recursive nature of integration by parts is one of the building blocks of modern mathematics. It is a feature of so many problems that it deserves consideration for inclusion into Paul Halmos' table of the mathematical elements.

- Accordingly,

  No doubt many mathematicians have noted that there are some basic ideas that keep cropping up in widely different parts of their subject, combining and re-combining with one another in a way faintly reminiscent of how all matter is made up of elements. A subconscious intuitive awareness of these "elements" of mathematics probably contributes to (possibly it constitutes) the research insight that distinguishes great mathematicians from ordinary mortals. – Paul Halmos[1]

- In this presentation, we examine the role of surface and volume and show how these are recurring themes tied to the derivative and integral.

We begin with a simple example, the circle and sphere

[1] Paul R. Halmos was born in Budapest, Hungary, on March 3, 1916, At the age of 15 he enrolled at the University of Illinois to study chemical engineering and later switched to mathematics and philosophy. He received his PhD in 1938. Halmos believed that mathematics is art, and that mathematicians are artists.

- Let S_n denote the surface area of an n-sphere, and V_n denote the volume or the content of an n-sphere.[2].
- The 2-sphere, i.e., the circle of radius R has circumference, i.e., surface $S_2 = 2\pi R$, and its area, or content is $V_2 = \pi R^2$.
- The surface area of a 3-sphere radius R is $S_3 = 4\pi R^2$, and its volume is $V_3 = (4/3)\pi R^3$.
- For the 2-sphere, $S_2 = \dfrac{d}{dr}V_2(r)$ since $\dfrac{d}{dr}\pi R^2 = 2\pi R$,
- For the 3-sphere, $S_3 = \dfrac{d}{dr}V_3(r)$ since $\dfrac{d}{dr}(4/3)\pi R^3 = 4\pi R^2$.

Does it generalize?

[2] Geometers call this the 2-sphere, topologists the 3-sphere

- If we assume that the volume of the n-sphere is proportional to the radius, then $V_n = \alpha_n r^n$, where α_n is our constant of proportionality for each dimension.
- Then by using concentric shells, $dV_n = S_n dr$, $\Longrightarrow$

$$\begin{aligned} S_n &= \frac{dV_n}{dr} \\ &= n\alpha_n r^{n-1}, \end{aligned}$$

and it becomes obvious why S_n is the derivative of V_n for each n-sphere.

We can do more. Let's evaluate S_n.

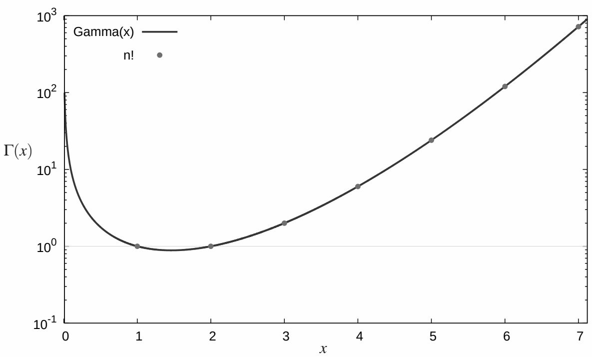

Figure: Plot of Gamma function, $\Gamma(x)$, showing factorials, $n! = x-1!$ for $x = 1, 2, \ldots$.

- The relationship between boundaries and volumes, and integrals and derivatives is expressed in $\mathbb{R}^n$ through the theorems of vector calculus, include Green's Theorem, the Divergence Theorem (Gauss's Theorem) and Stokes Theorem.
- These relate the integral of a vector quantity over a volume to the the integral of an associated vector quantity over a surface of the volume.
- For example, the Divergence Theorem has

$$\int_\Omega \nabla \cdot \boldsymbol{F} dV = \int_{\partial\Omega} \boldsymbol{F} \cdot \boldsymbol{n} dS,$$

where $\Omega \subset \mathbb{R}^3$ is a suitably defined domain with a piecewise smooth, oriented boundary, $\partial\Omega$.

We develop integration by parts in $\mathbb{R}^3$.

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